Fractals and nonstandard analysis

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We describe and analyze a parametrization of fractal "curves" (i.e., fractal of topological dimension 1). The nondifferentiability of fractals and their infinite length forbid a complete description based on usual real numbers. We show that using nonstandard analysis it is possible to solve this problem: A class of nonstandard curves (whose standard part is the usual fractal) is defined so that a curvilinear coordinate along the fractal can be built, this being the first step towards the possible definition and study of a fractal space. We mention fields of physics to which such a formalism could be applied in the future.

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I. INTRODUCTION

The concept of fractals, introduced by B. Mandelbrot,[1–3] applies to any curve, object, or set "whose form is extremely irregular and/or fragmented at all scales." More precisely, let D be the fractal dimension (e.g., the Hausdorff–Besicovitch dimension); a fractal is defined[2,3] as a set for which D is greater than the topological dimension D_T.

Nowadays this concept is increasingly considered in physics for several reasons:

(i) Fractal curves are functions which are continuous but nowhere differentiable; this property has already been observed for some natural phenomena, such as particle trajectories in quantum mechanics.[4,5]

(ii) The length of a fractal curve is dependent on the resolution with which it is measured and diverges when the resolution tends to be infinite.

(iii) A fractal dimension can be any real number, so this concept may apply to fields of physics such as critical behavior phenomena, where noninteger dimension has become a necessity.

Mandelbrot[1–3] pointed out many examples of the contribution fractals can bring to the description of natural phenomena such as the length of a coastline, the distribution of matter in the universe, turbulence, moon craters... . Furthermore, the concept of Haussdorff dimension has been applied to QCD jets,[6] gauge theories,[7] critical behavior,[8] fluctuations of the early universe,[9] or quantum-mechanical paths.[5]

However, in most cases, the authors limit themselves to the calculation of a fractal dimension or use fractals in a purely descriptive way (but see Le Mehauté *et al.*[10,11]). Because of the wide domain where phenomena seem to exhibit a fractal behavior, one is entitled to wish that a more thorough use of this concept would be possible, e.g., by building a formalism based on fractals and suitable for physics. In fact, no explicit calculation is presently possible on the fractal *itself* (i.e., the limit object instead of one of its approximations). It is the aim of this paper to show that nonstandard analysis, as built up by Robinson[12] is well adapted to such calculations.

In this paper, we first parametrize fractal "curves" (i.e., fractals of topological dimension 1) in the Cesaro[13,2] way (Sec. II). Then some paradoxical properties of fractals are evidenced and are clarified by the use of nonstandard analysis (Sec. III) as a way to build intrinsic curvilinear coordinates along a fractal curve (Sec. IV). This is hopefully a first step towards the definition of a fractal space by its own, while so far fractals have been considered as subsets of an integer-dimensional space.

II. PARAMETRIC EQUATION OF A FRACTAL CURVE

Consider a generalized von Koch curve in the $\mathbb{R}^2$ (or $\mathbb{C}$) plane. It can be built from an initial curve F_1 made up of p segments of equal length $1/q$ which connect the origin to the point [0,1] (see Fig. 1). Let ω_{j+1} be the polar angle of the jth segment and $Z_j = X_j + iY_j = (1/q)\sum_{k=0}^{j-1} e^{i\omega_k}$ the complex coordinate of a breaking point P_j. Two conditions hold between these data:

$$\sum_{k=0}^{p-1} e^{i\omega_k} = q, \quad Z_{j+1} - Z_j = \frac{1}{q} e^{i\omega_j}. \tag{2.1}$$

A curve F_2 is obtained by substituting each segment of F_1 by F_2 itself, scaled at its length $1/q$, as illustrated in Fig. 2. The resulting curve of an infinite sequence of these steps (substitution of each segment of F_n by $q^{-n} F_1$ giving F_{n+1}) is the fractal F.

As indicated by Mandelbrot[3] in the case of the Peano curve, F can be parametrized by a real number $x \in [0,1]$ developed in the counting base p in the form (see Fig. 2):

$$x = 0.x_1x_2 \cdots = \sum_1^\infty x_k\, p^{-k}. \tag{2.2}$$

The fractal will apparently be completely defined when the complex coordinate $Z(x)$ of the point on F parametrized by x is known. $Z(x)$ can be easily obtained, thanks to the above building process of F, under the form

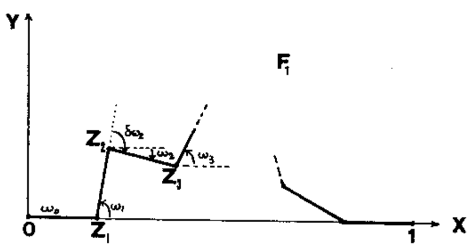

FIG. 1. Building of the basic structure of a fractal curve.

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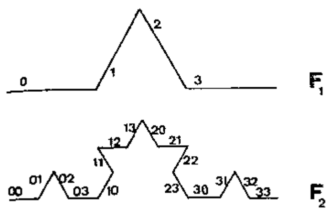

FIG. 2. Parametrization of a fractal curve.

$$Z(x) = Z_{x_1} + q^{-1} e^{i\omega_{x_1}} \left[Z_{x_2} + q^{-1} e^{i\omega_{x_2}} [Z_{x_3} + \cdots] \right], \tag{2.3}$$

so that we finally obtain

$$Z(x) = q \sum_{k=1}^{\infty} Z_{x_k} e^{i \Sigma_{l=1}^{k-1} \omega_{x_l}} q^{-k}. \tag{2.4}$$

If one defines new variables ρ_j, θ_j, and φ_{x_j} by writing

$$Z_j = \dot{q}\rho_j e^{i\theta_j}, \quad \varphi_{x_j} = \theta_{x_j} + \sum_{l=1}^{j-1} \omega_{x_l}, \tag{2.5}$$

another interesting form for the parametric equations of a fractal can be given

$$X(x) = \sum_{k=1}^{\infty} \rho_{x_k} \cos(\varphi_{x_k}) q^{-k},$$
$$Y(x) = \sum_{k=1}^{\infty} \rho_{x_k} \sin(\varphi_{x_k}) q^{-k}. \tag{2.6}$$

It is well known that, with its dimension lying between 1 and 2, a fractal curve within a plane is intermediate between a line and a surface. Indeed, while it may be built by adding segments it may also be obtained by deleting surfaces.[2,3] This construction allows dealing with the problem of multiple points. Given an initial curve F_1, consider a polygon P_0 of surface $\mathscr{S}_0$ with one of its diagonals being the segment [0,1] and in which F_1 is included. Then build around each segment of F_1 a q-reduced scale version of P_0: We obtain a figure P_1 (for example of such a construction, see Fig. 3). An obviously sufficient condition for the absence of multiple points is that all polygons of P_1 are disjointed, since this will remain true in the figures $P_2, P_3, \ldots, P_n \ldots$. Let $\delta\omega_j = \omega_j - \omega_{j-1}$ be the angle between two segments in F_1 (see Fig. 1). Then for two adjacent segments, the above condition leads to (angles α, β, γ are defined in Fig. 3)

$$\alpha^+ + \beta^+ < \pi - \delta\omega_j, \quad \alpha^- + \beta^- < \pi + \delta\omega_j, \tag{2.7}$$

so that we get for all values of j the sufficient conditions

$$-\gamma^- < \delta\omega_j < \gamma^+. \tag{2.8}$$

Let us now illustrate the nonstandard character of fractals, answering (by the negative) the question: Do Eqs. (2.2) and (2.6) characterize all properties of F in the $\mathbb{R}^2$ plane?

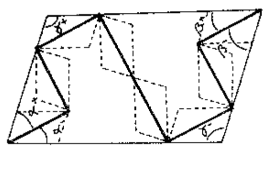

FIG. 3. Building of a fractal by deleting surfaces.

We first recall that, with its fractal dimension[2,3] $D = \log p/\log q$, the length of the fractal is infinite and its surface vanishes since the F_n length and surface are

$$\mathscr{L}_n / \mathscr{L}_0 = (p/q)^n = q^{n(D-1)},$$
$$\mathscr{S}_n / \mathscr{S}_0 = (p/q^2)^n = q^{-n(2-D)}. \tag{2.9}$$

Moreover, however small the difference of parameters $x(2) - x(1)$ for two points M_2 and M_1 on the fractal, though the distance in the plane $|Z(x_2) - Z(x_1)|$ vanishes, the distance along the fractal remains infinite.

However, let us build the following sequence:

$$a_n = a\mathscr{L}_n^{-1} = (aq^n)p^{-h} = 0.a_{n1}a_{n2} \cdots a_{np}, \tag{2.10}$$

where a is a nonzero number. Assume now that the a_n's are the parameters of a given sequence of points M_n on the curves F_n. The curvilinear coordinates of these points along F_n are equal to

$$l_n = \mathscr{L}_n a_n = a = \text{const}. \tag{2.11}$$

For example, in Fig. 4 the sequence of parameters 0.1, 0.03, 0.021, 0.0123, ... defines points at a constant distance 0.1 from the origin on the respective curves $F_1, F_2, F_3, F_4 \ldots$. Let now $n \to \infty$ and consider the limit point M of the sequence M_n on F. From Eq. (2.10) its parameter is $\lim(a_n) = 0$, i.e., *M coincides with the origin* 0; however, from Eq. (2.11), the curvilinear distance between O and M on the fractal is $a \neq 0$.

From this "paradox" we conclude that the real coordinate x is insufficient to describe thoroughly the fractal curve F: the distance along F between two points parametrized by two different x's is infinite, while points separated by a finite distance along F correspond to the same values of x. Thus another formalism is needed, and more precisely a set "larger" than $\mathbb{R}$: Nonstandard analysis[12] which allows dealing both with infinitesimals and infinite numbers, provides such a frame, well adapted to the study of fractals, as we show in the following sections.

III. NONSTANDARD ANALYSIS: A REMINDER

Nonstandard analysis (NSA) may be considered as the solution A. Robinson worked out for the old problem of infinitesimals. Leibnitz, founder of differential calculus, thought of "infinitely small" and "infinitely large" numbers as ideal numbers to which operations on usual numbers

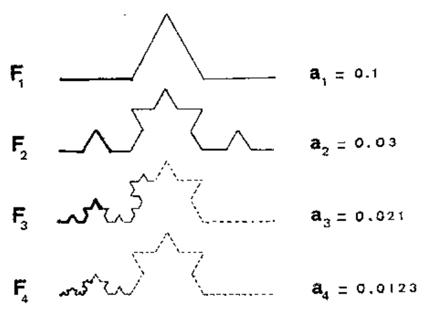

FIG. 4. A sequence of points on F_n with constant curvilinear distance to the origin.

would apply, though he was unable to build a coherent system: In fact the up to now accepted signification of these expressions involves the theory of limits and the so-called epsilon-delta method according to the works of Cauchy and Weierstrass. However, Robinson[12,14] has demonstrated that real numbers $\mathbb{R}$ can be extended to $^*\mathbb{R}$ which contains infinitely small and infinitely large numbers.

We will not try here to present a detailed description of NSA, but only recall some basic results useful in what follows independent of the precise way the theory is evolved, using, e.g., free ultrafilters and equivalence classes of sequences; see Robinson,[14] Stroyan and Luxemburg,[15] or using an axiomatic extension of the Zermelo set theory, see Nelson.[16]

The set $\mathbb{R}$ of hyper-real numbers is a totally ordered and non-Archimedean field. The set $\mathbb{R}$ of standard numbers is a subset of $^*\mathbb{R}$. $^*\mathbb{R}$ contains infinite elements, i.e, elements A such that $\forall n \in \mathbb{N}$, $|A| > n$. It also contains infinitesimal elements, i.e., B such that $\forall\ n \in \mathbb{N}\ (n \neq 0)$, $|B| < 1/n$. A finite element C is defined: $\exists\ m \in\ \mathbb{N}$, $|C| < m$. The set of infinitesimals is denoted by $\mathfrak{o}$, the set of finite numbers by $\mathfrak{O}$ and the set of infinite numbers by $^*\mathbb{R}_\infty$. Any finite number $a \in \mathfrak{O}$ can be split up in a single way as $a = r + \epsilon$, where $r \in \mathbb{R}$ and $\epsilon \in \mathfrak{o}$. In other words, the finite hyper–reals contain the ordinary reals with new numbers a clustered infinitesimally closely around each ordinary real r. Their set $\{a\}$ is called the monad of r. The real r is said to be the "standard part" of the hyper-real a, a function denoted by $r = \mathrm{st}(a)$. The "st" function is very useful for nonstandard demonstrations of standard theorems. For instance a sum $\Sigma_0^\infty f_n$ is said to converge if for different λ 's belonging to the set of infinite hypernatural numbers $^*\mathbb{N}_\infty$, st $(\Sigma_0^\lambda f_n)$ are all equal to the same finite number. Apart from the strict equality " = ", one introduces the equivalence relations "$\approx$" meaning "infinitely close to," i.e., $a \approx b\ \langle\Leftrightarrow\rangle \mathrm{st}(a-b) = 0$, and the relation "$\sim$" meaning "of the order of," i.e., $a \sim b\ \langle\Leftrightarrow\rangle\ \exists\ k \in \mathbb{R}$, $k \neq 0$ such that $a = kb$.

Formal descriptions of NSA may be found in Refs. 14–17. There have also been some attempts of applications of NSA to physics, e.g., Kelemen and Robinson,[18,19] Moore,[20] Anderson.[21]

IV. NONSTANDARD COORDINATES ALONG FRACTALS

In Sec. II, a fractal in $\mathbb{R}^2$ was parametrized by a real number belonging to the interval [0,1]:

$$x = \frac{x_1}{p} + \frac{x_2}{p^2} + \cdots + \frac{x_n}{p^n} + \cdots. \tag{4.1}$$

Let us generalize the usual fractal by introducing a curve F_ω in $^*\mathbb{R}^2$, parametrized by an hyper-real number $x^* \in {}^*[0,1]$, as defined by the *-finite power series expansion

$$x^* = \frac{x_1}{p} + \cdots + \frac{x_n}{p^n} + \cdots + \frac{x_{\omega-m}}{p^{\omega-m}} + \cdots + \frac{x_\omega}{p^\omega}, \tag{4.2}$$

where $\omega \in {}^*\mathbb{N}_\infty$. In other terms, F_ω may be obtained by applying the building process in Sec. II (i.e., build F_{n+1} by substituting to each segment of F_n, F_1 scaled by q^{-n}) ω times. It should be noticed that F_ω is *not* a fractal in the nonstandard sense (since the fragmentation is * limited up to ω) but its standard part is identical to the usual fractal, i.e.,

$$F \equiv \mathrm{st}(F_\omega). \tag{4.3}$$

Then the study of F_ω allows us to study the properties of F, thanks to the standardization axiom.

A first advantage is that, while the length of F was undefined, the length of F_ω is defined:

$$\mathscr{L}_\omega = (p/q)^\omega = q^{\omega(D-1)}. \tag{4.4}$$

While the surface of F was zero, the surface of F_ω is an infinitesimal:

$$\mathscr{S}_\omega = (p/q^2)^\omega = q^{-\omega(2-D)}. \tag{4.5}$$

The curvilinear coordinate ξ of the point parametrized by x^* is now also defined on F_ω:

$$\xi = x^* \mathscr{L}_\omega = q^{-\omega}\{x_\omega + x_{\omega-1}p + \cdots + x_1 p^{\omega-1}\}. \tag{4.6}$$

This verifies that F_ω is built up with elementary segments of length $q^{-\omega}$. By using an infinitely great magnifying power, F_ω can be drawn exactly (while this was not the case for F) as in Fig. 5. The fractal is no more a limit concept.

Let us now utilize the new concept of F_ω to study or clarify some problems specific of fractals, which may be relevant for physical applications.

A. Finite distance along the fractal

In Sec. II we obtained a point separated from the origin by a finite nonzero distance along the fractal, while its parameter was $x = 0$. This situation may be clarified by defining M such that

$$\xi_M = \mathscr{L}_\omega p^{-M} = 1. \tag{4.7}$$

Then M is a solution of the equation

$$p^{\omega-M} = q^\omega, \tag{4.8}$$

so that

$$M = \omega(1 - \log p/\log q) = \omega(1 - 1/D). \tag{4.9}$$

Generally M is not an integer; therefore we define $\lambda \in {}^*\mathbb{N}_\infty$ as

$$\lambda = \mathrm{Int}(M) = \mathrm{Int}[\omega(1 - 1/D)], \tag{4.10}$$

where $\mathrm{Int}(X)$ is the integer part of X (it is straightforward to

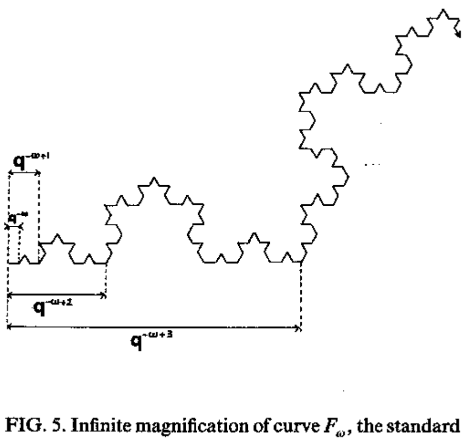

FIG. 5. Infinite magnification of curve F_ω, the standard part of which is the fractal F.

verify that this function is still defined in *R). For any λ' such that $\lambda' \sim \lambda$ (the relation $\sim$ "of the order of" has been defined in Sec. III), the curvilinear distance along the fractal $\xi_{\lambda'} = \mathscr{L}_\omega p^{-\lambda'}$ belongs to R. However, and this allows one to understand why $x = 0$ while $\xi \neq 0$, the corresponding distance in the $\mathbb{R}^2$ plane is an infinitesimal, $q^{-\lambda'}$, the standard part of which is thus zero.

Consider two points on F_ω separated by a curvilinear distance ξ. Depending on the power n of p in the expression (4.6) of ξ, three levels may be distinguished on F_ω:

—$n \in \mathbb{N}$: Finite distance in the $\mathbb{R}^2$ plane, infinite along the fractal.

—$n \sim \lambda$: Infinitesimal distance in the $\mathbb{R}^2$ plane, finite distance along the fractal.

—$n \sim \omega$: Infinitesimal distance in $\mathbb{R}^2$ and on F_ω. At this level the two distances are of the same order.

B. Intrinsic building of F

Parametric equations for F have been given in Sec. II (Eq. 2.6): It relates the parameter x to the coordinates X, Y in the $\mathbb{R}^2$ plane. An intrinsic building of F_ω, independent of the plane in which it is embedded, is possible: We only need to know the change in direction from each elementary segment of length $q^{-\omega}$ to the following one. In Sec. II, we had set $\delta\omega_j = \omega_j - \omega_{j-1}$, the angle between two segments in F_1 (see Fig. 1). Consider the infinitesimal segment of curvilinear coordinate ξ on F_ω, and define h such that $x_{\omega-h}$ is the first nonzero figure of the hypernatural number $q^\omega \xi$ in the base p:

$$\xi = q^{-\omega} \times \{0 + 0 \times p + \cdots + 0 \times p^{h-1} + x_{\omega-h} p^h + \cdots + x_1 p^{\omega-1}\}. \tag{4.11}$$

The relation for the angle we were looking for and which allows an intrinsic building of F_ω is simply given by

$$\delta\omega_j(\xi) = \delta\omega_{x_{\omega-h}}. \tag{4.12}$$

A somewhat paradoxical property of fractals is that this relative angle exists and may be computed for any value of ξ, while on the contrary the absolute angle $\omega_j(\xi)$ does not exist as soon as ξq^ω becomes infinite, since

$$\omega(\xi) = \sum_{q^\omega \xi' = 1}^{q^\omega \xi} \delta\omega_j(\xi') \tag{4.13}$$

so that the value of st $[\omega(\xi)]$ depends on that of ω (this is the nonstandard transfer of the nondifferentiability of F).

C. Family of curves F_ω and ϵ differentiability

The standard part of any curve differing from F_ω only by infinitesimals will also be the fractal F. This is true for any $F_{\omega'}$ with $\omega' \in {}^*\mathbb{N}_\infty$, $\omega' \neq \omega$, but might be generalized. In particular the "broken" aspect of F_ω can be given up and F_ω replaced by a smoothed nonstandard curve F'_ω, with $F = \mathrm{st}(F'_\omega)$. A kind of differentiability can be defined for F'_ω.

Indeed, nonstandard analysis may be used to define differentiability of standard functions[15]:

$$\forall\ x \approx y \approx a, \frac{f(x) - f(a)}{x - a} \approx \frac{f(y) - f(a)}{y - a} \text{ finite,}$$

then $f'(a) = \mathrm{st}\left(\dfrac{f(x) - f(a)}{x - a}\right)$.

This definition could be generalized to an "ϵ differentiability" of nonstandard curves in the following way: If $\exists\, \epsilon \in \mathfrak{o}$, such that $\forall\, x, y, a \in {}^*\mathbb{R}$ verifying $|x - a| < \epsilon$ and $|y - a| < \epsilon$, and

$$\frac{f(x) - f(a)}{x - a} \approx \frac{f(y) - f(a)}{y - a}$$

finite, then

$$f'_\epsilon(a) = \mathrm{st}\left(\frac{f(x) - f(a)}{x - a}\right).$$

In that sense, one can define a curve F'_ω which is ϵ differentiable everywhere, since the above definition is verified by taking $\epsilon = \epsilon_0$ with $\epsilon_0 q^\omega \approx 0$. It is clear that differentiability implies ϵ differentiability but that the reverse proposition is false, as shown by fractals.

V. GENERALIZATIONS AND CONCLUSION

To simplify the description of the concepts introduced in this paper, we have worked only with regular fractals of topological dimension 1 in $\mathbb{R}^2$. Some straightforward generalizations may be considered.

For instance a fractal curve in $\mathbb{R}^3$ is parametrized in the same way as in $\mathbb{R}^2$, the angles ω_j of the elementary segments of F_1 being replaced by rotation matrices R_j that define the new reference system tied to each segment $\mathbf{v}_j$; the elementary points are defined by vectors u_j, with $\mathbf{v}_j = \mathbf{u}_{j+1} - \mathbf{u}_j$. Thus, if p and q have the same definitions as above, a point of parameter $x = 0.x_1x_2 \cdots x_n \cdots$ in the counting base p is referenced by a vector:

$$\mathbf{u}(x) = \sum_{n=0}^{\infty} R_{x_n} R_{x_{n-1}} \cdots R_{x_1} \mathbf{u}_{x_{n+1}} q^{-n}. \tag{5.1}$$

Such a curve has a topological dimension 1 and a fractal dimension $D = \log p / \log q$ lying between 1 and 3.

Another possible generalization is to give up self-similarity and to build a more general fractal curve in $\mathbb{R}^2$. F_1 is built up with p_0 segments, the coordinates of the "breaking points" being Z_{x_1}, with $x_1 = 0, 1, \ldots, p - 1$. In each segment of F_1 referenced by x_1, we introduce p_{x_1} new segments defined by $Z_{x_1x_2}$, $x_2 = 0, 1, \ldots, p_{x_1} - 1$, and angle $\omega_{x_1x_2}$; we then build F_2. This construction is extended to the infinite, by giving ourselves the coordinates $Z_{x_1 \cdots x_n}$, where $x_n \in \{0, 1, \ldots, p_{x_1 \cdots x_{n-1}} - 1\}$ and angles $\omega_{x_1 \cdots x_n}$. There are $p_{x_1 \cdots x_{n-1}}$ new segments of length $1/q_{x_1 \cdots x_n}$. Since the base p varies from segment to segment, the sequence $x_1x_2 \cdots x_n \cdots$ is now only a code and cannot be summarized by a number. The complex coordinate in the $\mathbb{R}^2$ plane of a point on F is then

$$Z(x_1x_2 \cdots x_n \cdots) = \sum_{n=0}^{\infty} \frac{Z_{x_1 \cdots x_{n+1}} e^{i(\omega_{x_1} + \cdots + \omega_{x_1 \cdots x_n})}}{q_{x_1} q_{x_1x_2} \cdots q_{x_1 \cdots x_n}}. \tag{5.2}$$

This construction is obviously not unique. The dimension of F could be defined as $\langle p \rangle \langle q^{-D} \rangle = 1$ (see Mandelbrot[2]).

Up to now, only fractals with a globally given dimension D have been discussed. However the fractal dimension can clearly be a local property,[2] varying with the curvilinear coordinate $D = D(\xi)$; this is more easily understandable by remembering the underlying infinitesimal structure.

In the future, we will try to extend the parametrization presented here to fractals of topological dimensions greater than one. Instead of studying fractal *objects* embedded in an Euclidean space, the aim of such a work would be to define a *fractal space* intrinsically. Its dimension could then be a generic parameter and function of the coordinates in the same way as curvature occurs for curved spaces.

In a forthcoming paper,[22] one of the authors will consider physical applications of this formalism to quantum mechanics (uncertainty relations and theory of measure). We think the notions developed here could help in the field of quantization of gravity: the studies in this field usually assume without question an underlying, eventually foamy,[23] 4-manifold. Below some characteristic length, space itself could become a fractal. To be more specific one could for instance perform latticelike gauge field calculations[24] on a fractal. Let us notice that the path-integral ingredient of such calculations has also recently been formulated in terms of nonstandard analysis.[20] When applied to cosmology these ideas lead to the natural speculation that the very early universe experienced in its whole a noninteger-dimensional phase. To conclude we hope that these new trends will help to answer the question: Why does the macroscopic space appear to be three dimensional?

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